Fixed Points and 2-Cycles of Synchronous Dynamic Coloring Processes on Trees

Volker Turau

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Institute of Telematics
Hamburg University of Technology
TUHH

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Introduction

Dynamical Systems for Information Spreading

- Epidemic spreading based on networks is researched in several disciplines
  - Nodes represent individuals and links denote interactions
  - An *item* is transmitted from one individual to another through link between them
- Network structure is an important factor for efficiency of epidemic spreading
- Various models
  - Deterministic vs. probabilistic
  - Synchronous vs. asynchronous
  - Stationary vs. mobile
  - Discrete vs. continuous

Discrete Synchronous Dynamical Systems (DSDS)

- In discrete-time rounds, all nodes concurrently update their state based on local rule
  - Node's state in round t is determined by states of neighboring nodes in round $t-1$
- Facts
  - After a finite number of rounds a DSDS either reaches fixed point or enters 2-cycle
  - Finding number of fixed points of a DSDS is in general #P-complete
- Classic research topics
  - Stabilization time: After how many rounds does system become stable
  - Dominance problem: Which initial states lead to a homogeneous system state

Fixed Points and 2-Cycles

- Model of this work
  - State of a node is a color: 0 or 1
  - Local rules: Majority and minority rule
  - Finite trees

- Our main contributions
  - Simple graph-theoretic characterization of fixed points and 2-cycles
  - Upper bounds for number of fixed points and 2-cycles
  - Algorithm to generate all fixed points and 2-cycles

Motivation

- Boolean networks (BN) model the dynamics of gene regulatory networks
- BNs are special type of DSDS for the majority rule
- Number of fixed points is a measure for general memory storage capacity of BN
- BNs can solve SAT problems
- BN fixed points correspond to SAT solutions

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Definitions

Definitions

Discrete Synchronous Dynamical Systems (DSDS)

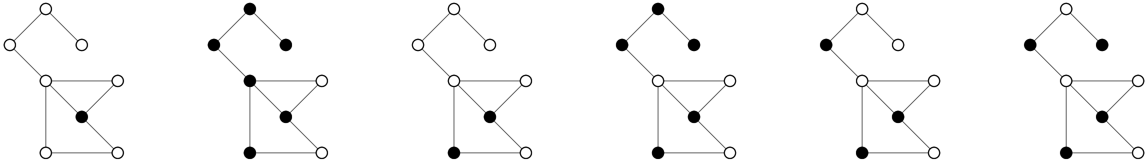
Let $G(V,E)$ be a finite graph. A **coloring** c assigns to each node a value of $\{0,1\}$. $\mathcal{C}(G)$ denotes the set of all colorings of G.

A **DSDS** is a mapping $\mathcal{M} : \mathcal{C}(G) \longrightarrow \mathcal{C}(G)$. For an initial coloring c, $\mathcal{M}$ yields a series of colorings $c, \mathcal{M}(c), \mathcal{M}(\mathcal{M}(c)), \ldots$.

Minority Process

Minority process $\mathcal{MIN}$: Each node assumes the color of the minority of its neighbors.

Example of Minority Process



- Minority process reaches fixed point after five rounds

Specific Colorings

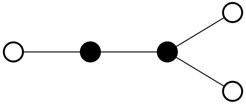
Fixed Points and 2-Cycles

$c \in \mathcal{C}(G)$ is called

- **fixed point** if $\mathcal{M}(c) = c$
- **2-cycle** if $\mathcal{M}(c) \neq c$ and $\mathcal{M}(\mathcal{M}(c)) = c$
- **monochromatic** if all nodes have the same color
- **independent** if the color of each node is different from the colors of all its neighbors

A 2-cycle c is called **pure** if it is $\mathcal{M}(c)(v) \neq c(v)$ for each node v of G

Examples



A fixed point

A 2-cycle

Classes of Colorings

Classes of Colorings

- $\mathcal{F}_{\mathcal{M}}(G)$: All $c \in \mathcal{C}(G)$ that constitute a fixed point
- $\mathcal{C}^2_{\mathcal{M}}(G)$: All $c \in \mathcal{C}(G)$ that constitute a 2-cycle
- $\mathcal{P}_{\mathcal{M}}(G)$: All $c \in \mathcal{C}(G)$ that constitute a pure coloring

- Classes are closed with respect to complements
- Let $\mathcal{F}_{\mathcal{M}}(G)^+$, $\mathcal{C}^2_{\mathcal{M}}(G)^+$, and $\mathcal{P}_{\mathcal{M}}(G)^+$ be the subsets of those colorings of corresponding sets which assign to a globally distinguished node v^* color 0

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Fixed Points of Trees

Characterizing Fixed Points

- Let $T = (V, E)$ be a tree
- Goal: Characterization of $\mathcal{F}_{\mathcal{M}}(T)$
- We identify a nonempty subset $E_{fix}(T) \subset 2^E$ and define a bijection

$$\mathcal{B}_{fix} : E_{fix}(T) \longrightarrow \mathcal{F}_{\mathcal{M}}(T)^+$$

- It is easy to compute $E_{fix}(T)$

Characterizing Fixed Points

- Let $v \in V$ and $F \subseteq E$. Then F_v denotes the number of edges of F incident with v

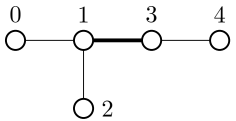
Definition

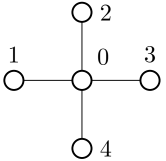
Let $E^2(T)$ be the set of edges, where each end node has degree at least 2. $F \subseteq E^2(T)$ is called **legal** if $F_v \leq deg(v)/2$ for each node v.

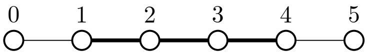
$E_{fix}(T)$ denotes the set of all legal subsets F of $E^2(T)$

Note that $E_{fix}(T)$ satisfies the hereditary property

Characterizing Fixed Points







- $E_{fix}(T_1) = \{\emptyset, \{(1,3)\}\}$
- $E_{fix}(T_2) = \{\emptyset\}$
- $E_{fix}(T_3) = \{\emptyset, \{(1,2)\}, \{(2,3)\}, \{(3,4)\}, \{(1,2),(3,4)\}\}$

Characterizing Fixed Points

Theorem

For each tree T there exists a bijection $\mathcal{B}_{fix}$ between $E_{fix}(T)$ and $\mathcal{F}_{\mathcal{M}}(T)^+$.

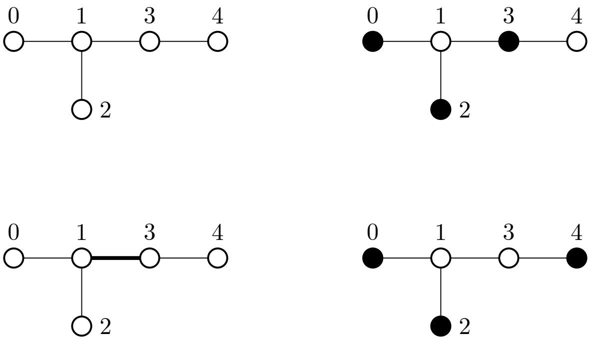
General idea of proof.

- For $F \in E_{fix}(T)$ define $c_F \in \mathcal{F}_{\mathcal{MIN}}(T)^+$ such that edges in F are monochromatic
- Let $\mathcal{C}_T(F)$ be the set of connected components of $T \setminus F$
- Let $T^* \in \mathcal{C}_T(F)$ with $v^* \in T^*$. Let $c_F(v^*) = 0$
- Extend c_F to an independent coloring of T^* (breadth-first search)
- While there exists an already colored node u with an uncolored neighbor w in T do
  - Let $T_1 \in \mathcal{C}_T(F)$ with $w \in T_1$
  - Let $c_F(w) = c_F(v)$ and extend c_F to an independent coloring of T_1 (tricky!)

□

Characterizing Fixed Points

- Examples for $F = \emptyset$ and $F = \{(1,3)\}$



Counting Fixed Points of Trees

Corollary

Every tree has at least two fixed points.

Theorem

Let T be a tree. Then $|\mathcal{F}_{\mathcal{M}}(T)| \leq 2F_{n-\lceil \Delta/2 \rceil}$. If T is a path then $|\mathcal{F}_{\mathcal{M}}(T)| = 2F_{n-1}$.

- Bound is not sharp (star graph)

Generating Fixed Points of Trees

Theorem

There exists an algorithm to compute all $|\mathcal{F}(T)|$ fixed points of a tree T in time $O(n + |\mathcal{F}(T)||E^2(T)|)$ using $O(|E^2(T)||\mathcal{F}(T)|)$ memory.

- Algorithm exploits hereditary property of $E_{fix}(T)$

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Pure 2-Cycles

Pure 2-Cycles

- Similar type of characterization

Definition

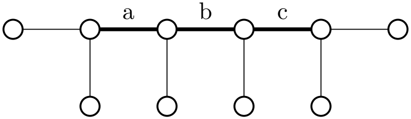
Let $E^3(T)$ be the set of edges, where each end node has degree at least 3. $F \subseteq E^3(T)$ is called **legal** if $F_v < deg(v)/2$ for each node v.

$E_{pure}(T)$ denotes the set of all legal subsets F in $E^3(T)$.

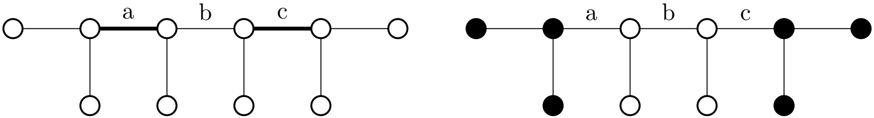
Theorem

For each tree T there exists a bijection $\mathcal{B}_{pure}$ between $E_{pure}(T)$ and $\mathcal{P}_{\mathcal{M}}(T)^+$.

Characterizing Pure 2-cycles of Trees



- $E^3(T)$ consists of three edges (solid lines)
- $E_{pure}(T_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}\}$
- Pure 2-cycle for $F = \{a, c\}$



Counting Pure 2-cycles of Trees

- $|\mathcal{P}_{\mathcal{M}}(T)| \leq 2^{1+(n-4)/2}$
- Bound is not sharp (see previous example)
- Efficient algorithm for generating pure 2-cycles

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Block Trees of 2-Cycles

Block Tree

Definition

Let T be a tree and $c \in \mathcal{C}^2_{\mathcal{M}}(T)$. Let V_f (resp. V_t) be the set of fixed (resp. toggle) nodes of c and T^f (resp. T^t) the subgraph of T induced by V_f (resp. V_t).

Lemma

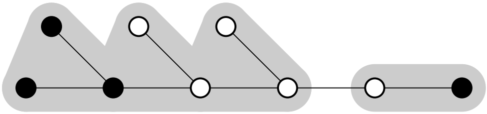
Let T be a tree, $c \in \mathcal{C}^2_{\mathcal{M}}(T)$, and T' a connected component of T^f (resp. T^t). Then c induces a fixed point (resp. a pure 2-cycle) on T'.

Block Tree

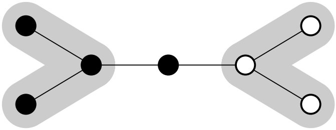
Definition

Let T be a tree, $c \in \mathcal{C}^2_{\mathcal{M}}(T)$, and $T_1, \ldots, T_s$ the connected components of T^f and T^t. The **block tree** $\mathcal{B}_c(T)$ of T for c is a tree with nodes $\{T_1, \ldots, T_s\}$. T_i and T_j are connected if there exists $(u, w) \in E$ with $u \in T_i$ and $w \in T_j$.
A node T_i is called a *fixed block* (resp. *toggle block*) of $\mathcal{B}_c(T)$ if T_i is a connected component of T^f (resp. T^t).

Block Tree



One toggle and one fixed block



Two toggle and one fixed block

Block Tree

Definition

$E^{2.5}(T)$ denotes the set of edges, where one end node has degree at least 2 and the other at least 3. For $F \subseteq E^{2.5}(T)$ a component $\hat{T} \in \mathcal{C}_T(F)$ is called *fixed* if $|\hat{T}| = 1$ or if there exists $v \in \hat{T}$ such that $deg_T(v) \equiv 0(2)$ and $deg_{\hat{T}}(v) = 1$. $Fix(T, F)$ denotes the set of all fixed components of $\mathcal{C}_T(F)$.

$F \subseteq E^{2.5}(T)$ is called **legal** if all components of $Fix(T, F)$ are fully contained in $\mathcal{I}_0(\mathcal{T}_F)$ and if $T_0 \in \mathcal{C}_T(F)$ with $T_0 = \{v\}$ then $deg_T(v) \equiv 0(2)$.

$E_{block}(T)$ denotes the set of all legal subsets of $E^{2.5}(T)$.

Block Tree

Theorem

For each tree T there exists a bijection $\mathcal{B}_{block}$ between $E_{block}(T)$ and the block trees of T.

- Note, $E_{block}(T)$ does not satisfy the hereditary property
- No efficient algorithm to generate all block trees is known

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Conclusion

Conclusion & Outlook

- Contributions
  - Characterization of fixed points, pure and general 2-cycles for minority/majority process
  - Algorithm to enumerate all fixed points and pure 2-cycles
  - Upper bounds for number of fixed points
- Open problems
  - Generalization to other graph classes (results do not hold for cycles)
  - Better upper bounds
  - Compute expected sizes for random trees

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29th Int. Colloquium on Struct … plexity

Volker Turau

Professor

Phone +49 / (0)40 428 78 3530

e-Mail turau@tuhh.de

http://www.ti5.tu-harburg.de/staff/turau