Sharp Upper Bounds for Reconfiguration Sequences of Independent Sets in Trees

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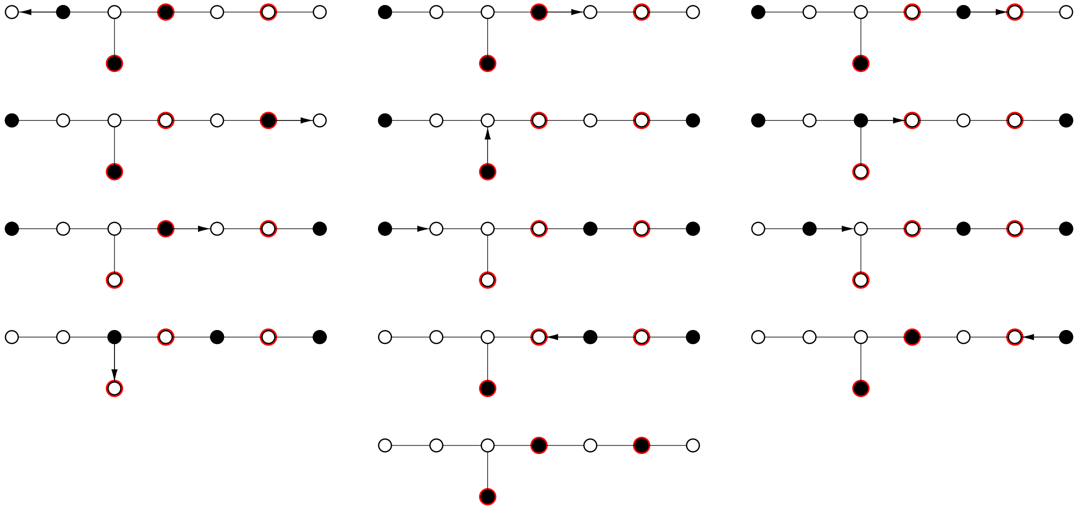
Reconfiguration Sequences of Independent Sets

- $T = (V, E)$ finite tree with n nodes, $s \in \mathbb{N}$ with $1 \leq s \leq n - 1$
- $\mathcal{I}_s(T)$ set of independents subsets of T of size s
- *Reconfiguration graph*
  - Undirected graph $\mathcal{G}_s(T)$ with node set $\mathcal{I}_s(T)$
  - $S_1, S_2 \in \mathcal{I}_s(T)$ are connected by an edge if there exists $(v, w) \in E$ s.t. $v \in S_1 \setminus S_2, w \in S_2 \setminus S_1$, and $S_2 = S_1 \setminus \{v\} \cup \{w\}$
- **Token sliding model**
- Question: What is maximal diameter $\mathcal{RD}(n, s)$ of $\mathcal{G}_s(T)$ for given n, s?

Example: $\mathcal{RD}(8,3) = 12$



Known Results

- Distances in reconfiguration graphs can be computed in $O(n)$ time for cographs, claw-free graphs, and trees [Demaine et al. 2015]
- Diameter of reconfiguration graphs for these classes is in $O(n^2)$
- Family of instances on paths for which diameter of reconfiguration graph is in $\Omega(n^2)$

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Results & Conjectures

Regions

- Value of $\mathcal{RD}(n, s)$ depends on ratio between s and n
- We consider three regions:

1. $s \ll n$
2. $s > n/2$
3. region in between

Region $s \ll n$

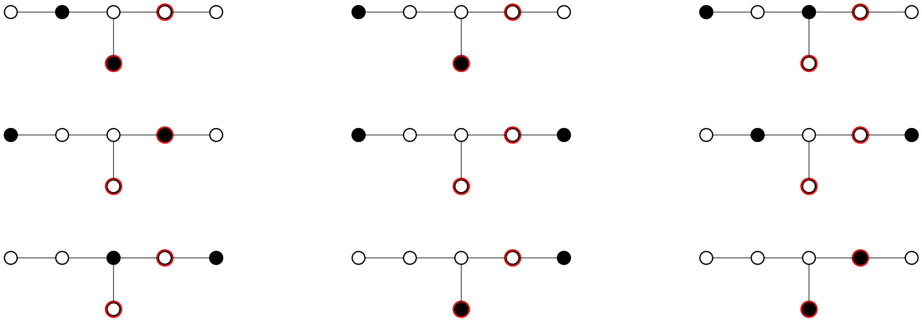
- To maximize $\mathcal{RD}(n,s)$ we need to place S_1 and S_2 as far appart as possible
- If s is small with respect to n then this is best possible if T is a path

Theorem

$\mathcal{RD}(n,s) = s(n-2s+1)$ *for* $4s < n+2$.

- Lemma does not hold for $n = 6, s = 2$

Example: $\mathcal{RD}(6,2) = 8$



Region $s > n/2$

- Observation: $\mathcal{RD}(n, s) \leq \mathcal{RD}(n + i + 1, s + i)$ for $i \geq 2$
  - Attach to any node of T a star T_i with i nodes, place leaves of T_i into S_1 and S_2 (provide picture)

Lemma

Let $2s + 1 \geq n$. If T, S_1, S_2 realize $\mathcal{RD}(n, s)$ then there exists a node v with degree $i \geq 2$ such that all but one neighbor of v are leaves contained in $S_1 \cap S_2$.

- If $2s + 1 \geq n$ there exists $i \geq 2$ such that $\mathcal{RD}(n, s) = \mathcal{RD}(n - i, s - (i - 1))$.

Theorem

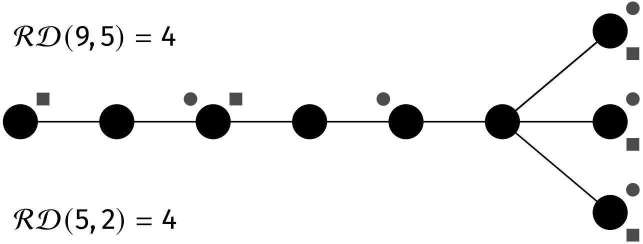
There exists a sequence C_i of numbers such that $\mathcal{RD}(n, s) = C_{n-s}$ for $2s + 1 \geq n$.

Region $s > n/2$

- Conjecture

$n-s$	$\mathcal{RD}(n,s)$
2	1
3	2
4	4
5	8
6	12
7	21

- $\mathcal{RD}(9,5) = 4$
- $\mathcal{RD}(5,2) = 4$

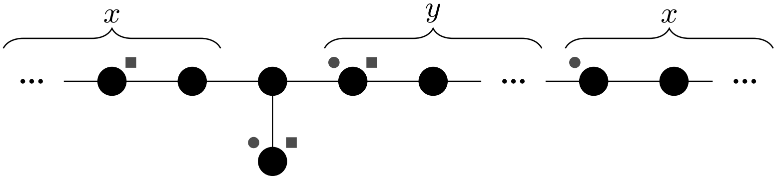


Intermediate Region: $4s > n > 2s + 1$

n	$\mathcal{RD}(n,5)$	$\mathcal{RD}(n,6)$	$\mathcal{RD}(n,7)$
12	21		
13	22		
14	30	26	
15	30	27	
16	40	36	31
17	40	36	32
18	50	48	43
19	50	48	44
20		60	56
21		60	56
22		72	70
23		72	70
24			84
25			84

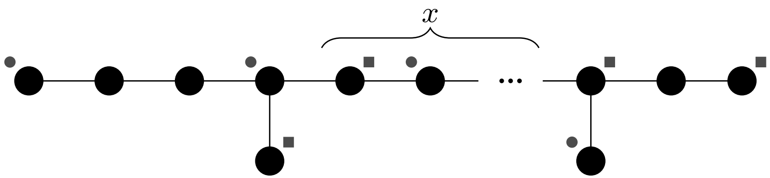
- No obvious pattern
- No generic tree found
- Region is covered by trees of diameter $n-2$ and $n-3$

Type 1



- $s = x + y + 1$, $n = 4x + 2y + 2$

Type 2



- $s = x + 2$, $n = 9 + 2x$

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Conclusion

Conclusion & Outlook

- Challenge: Maximal diameter $\mathcal{RD}(n, s)$ of $\mathcal{G}_s(T)$ for given n, s
- Three regions identified
- Some cases are still open

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